INVARIANCE PROPERTIES OF THEMATIC FACTORIZATIONS OF MATRIX FUNCTIONS

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Abstract. We study the problem of invariance of indices of thematic factorizations. Such factorizations were introduced in [PY1] for studying superoptimal approximation by bounded analytic matrix functions. As shown in [PY1], the indices may depend on the choice of a thematic factorization. We introduce the notion of a monotone thematic factorization. The main result shows that under natural assumptions a matrix function that admits a thematic factorization also admits a monotone thematic factorization and the indices of a monotone thematic factorization are uniquely determined by the matrix function itself. We obtain similar results for so-called partial thematic factorizations.

1. Introduction

It is well known [Kh] that for a continuous scalar function φ on the unit circle $\mathbb{T}$ there exists a unique function $f \in H^\infty$ such that

$$\operatorname{dist}_{L^\infty}(\varphi, H^\infty) = \|\varphi - f\|_{L^\infty}.$$

However, the situation in the case of matrix-valued function is considerably more complicated.

Suppose that Φ is a matrix function in $L^\infty(\mathbb{M}_{m,n})$, i.e., Φ is an essentially bounded function on the unit circle $\mathbb{T}$ that takes values in the space $\mathbb{M}_{m,n}$ of $m \times n$ matrices. We say that a function $F \in H^\infty(\mathbb{M}_{m,n})$ (by this we mean that all entries of F belong to H^∞) is a *best approximation of* Φ by bounded analytic matrix functions if

$$\|\Phi - F\|_{L^\infty} = \operatorname{dist}_{L^\infty}(\Phi, H^\infty(\mathbb{M}_{m,n})).$$

Here for a function Ψ in $L^\infty(\mathbb{M}_{m,n})$ we use the notation

$$\|\Psi\|_{L^\infty} \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} \|\Psi(\zeta)\|_{\mathbb{M}_{m,n}},$$

where $\mathbb{M}_{m,n}$ is equipped with the operator norm from $\mathbb{C}^n$ to $\mathbb{C}^m$.

The second author is partially supported by NSF grant DMS 9970561.

It is easy to see that unlike the scalar case we can have uniqueness only in exceptional cases. Indeed, if $\Phi = \begin{pmatrix} \bar{z} & 0 \\ 0 & 0 \end{pmatrix}$, then $\operatorname{dist}_{L^\infty}(\Phi, H^\infty(\mathbb{M}_{2,2})) = 1$ since $\operatorname{dist}_{L^\infty}(\bar{z}, H^\infty) = 1$. Clearly, for any scalar function $f \in H^\infty$ such that $\|f\|_\infty \le 1$ we have

$$\left\| \begin{pmatrix} \bar{z} & 0 \\ 0 & -f \end{pmatrix} \right\|_{L^\infty} = 1,$$

and so $\begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix}$ is a best approximation of Φ.

Recall that by a matrix analog of Nehari's theorem (see [Pa]),

$$\operatorname{dist}_{L^\infty}(\Phi, H^\infty(\mathbb{M}_{m,n})) = \|H_\Phi\|,$$

where the *Hankel operator* $H_\Phi : H^2(\mathbb{C}^n) \to H^2_-(\mathbb{C}^m)$ is defined by

$$H_\Phi f \stackrel{\text{def}}{=} \mathbb{P}_- \Phi f, \quad f \in H^2(\mathbb{C}^n).$$

Here $\mathbb{P}_-$ is the orthogonal projection onto $H^2_-(\mathbb{C}^m) \stackrel{\text{def}}{=} L^2(\mathbb{C}^m) \ominus H^2(\mathbb{C}^m)$.

Recall also that by Hartman's theorem (see e.g., [N]), H_Φ is compact if and only if $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$, where

$$H^\infty + C \stackrel{\text{def}}{=} \{f + g : \ f \in H^\infty, \ g \in C(\mathbb{T})\}.$$

(Throughout the paper we write $\Phi \in X(\mathbb{M}_{m,n})$ if all entries of an $m \times n$ matrix function Φ belong to a function space X; sometimes to simplify the notation we will write simply $\Phi \in X$ if this does not lead to a confusion.)

In [PY1] it was shown that if $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$, then there exists a unique function $F \in H^\infty(\mathbb{M}_{m,n})$ that minimizes (lexicographically) not only $\|\Phi - F\|_{L^\infty}$ but also the essential *suprema*

$$t_j \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} s_j(\Phi(\zeta) - F(\zeta)), \quad j \le \min\{m, n\} - 1$$

of all subsequent singular values of $\Phi(\zeta) - F(\zeta)$, $\zeta \in \mathbb{T}$. Such functions F are called *superoptimal approximations of* Φ by bounded analytic matrix functions. The numbers t_j are called *superoptimal singular values of* Φ. It was also shown in [PY1] that the error function $\Phi - F$ admits certain special factorizations (*thematic factorizations*). For each such factorization the sequence of positive indices k_j, $j \ge 0$, $t_j > 0$, (*thematic indices*) was defined. We refer the reader to §2 where formal definitions are given. Note that another approach to superoptimal approximation was found later in [T].

In [PT2] the same results were proved for functions $\Phi \in L^\infty(\mathbb{M}_{m,n})$ such that the essential norm $\|H_\Phi\|_{\mathrm{e}}$ of H_Φ (i.e., the distance from H_Φ to the set of compact operators) is less than the smallest nonzero superoptimal singular value of Φ. Recall

that

$$\|H_\Phi\|_{\rm e} = \operatorname{dist}_{L^\infty}\big(\Phi, (H^\infty + C)(\mathbb{M}_{m,n})\big)$$

(see e.g., [S] for the proof of this formula for scalar functions, in the matrix-valued case the proof is the same).

It turned out, however, that the thematic indices are not uniquely determined by the function Φ itself but may depend on the choice of a thematic factorization (see [PY1]). On the other hand it was shown in [PY2] (see also [PT2]) that the sum of the thematic indices that correspond to the superoptimal singular values equal to a specific number is uniquely determined by Φ.

In this paper we show that one can always choose a so-called monotone thematic factorization, i.e., a thematic factorization such that the indices that correspond to equal superoptimal nonzero singular values are arranged in the nonincreasing order. We refer the reader to §4 for a formal definition. We prove in §3 and §4 that the indices of a monotone thematic factorization are uniquely determined by the function Φ itself. Section 2 contains definitions and statements of basic results on superoptimal approximation and thematic factorizations.

Note that using the same methods we can obtain similar results in the case of the four block problem (which is an important generalization of the problem of best approximation by bounded analytic matrix functions). We refer the reader to [PT2] which contains results on superoptimal approximation and thematic factorizations related to the four block problem.

We can also obtain similar results in the case of infinite matrix functions. We refer the reader to [T], [Pe], and [PT1] for results on superoptimal approximation and thematic factorizations for infinite matrix functions.

2. Superoptimal approximation and thematic factorizations

In this section we collect necessary information on superoptimal approximation and thematic factorizations.

Let $\Phi \in L^\infty(\mathbb{M}_{m,n})$. We put

$$\Omega_0 = \{F \in H^\infty(\mathbb{M}_{m,n}) : \ F \text{ minimizes } t_0 \stackrel{\rm def}{=} \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} \|\Phi(\zeta) - F(\zeta)\|\};$$

$$\Omega_j = \{F \in \Omega_{j-1} : \ F \text{ minimizes } t_j \stackrel{\rm def}{=} \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} s_j(\Phi(\zeta) - F(\zeta))\}.$$

Recall that for a matrix $A \in \mathbb{M}_{m,n}$ the *jth singular value* $s_j(A)$ is defined by

$$s_j(A) = \inf\{\|A - R\| : \ \operatorname{rank} R \le j\}, \quad j \ge 0.$$

Functions F in $\Omega_{\min\{m,n\}-1}$ are called *superoptimal approximations of* Φ *by analytic functions*, or superoptimal solutions of the Nehari problem. The numbers t_j

are called *superoptimal singular values of* Φ. The notion of superoptimal approximation plays an important role in H^∞ control theory.

It can be shown easily with the help of a compactness argument that the sets Ω_j are nonempty. In particular, for any matrix function in $L^\infty(\mathbb{M}_{m,n})$ there exists a superoptimal approximation by analytic matrix functions.

It was shown in [PY1] that for any matrix function $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$ there exists a unique superoptimal approximation. We denote by $\mathcal{A}\Phi$ the unique superoptimal approximation of Φ by bounded analytic matrix functions whenever it is unique.

Later in [PT2] stronger results were obtained. It was shown there that if $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and the essential norm $\|H_\Phi\|_{\mathrm{e}}$ of the Hankel operator H_Φ is less than the smallest nonzero superoptimal singular value of Φ, then Φ has a unique superoptimal approximation by bounded analytic matrix functions.

A matrix function $\Phi \in L^\infty(\mathbb{M}_{m,n})$ is called *badly approximable* if

$$\operatorname{dist}_{L^\infty}(\Phi, H^\infty(\mathbb{M}_{m,n})) = \|\Phi\|_{L^\infty}.$$

It is called *very badly approximable* if the zero matrix function is a superoptimal approximation of Φ.

Recall that a nonzero scalar function $\varphi \in H^\infty + C$ is badly approximable if and only if it has constant modulus almost everywhere on $\mathbb{T}$, belongs to QC, and its winding number $\operatorname{wind}\varphi$ is negative, where the space QC of quasi-continuous functions is defined by

$$QC = \{f \in H^\infty + C : \ \bar{f} \in H^\infty + C\}.$$

For continuous φ this was proved in [Po] (see also [AAK1]). For the general case see [PK]. Recall that if $\varphi \in QC$ and φ has constant modulus on $\mathbb{T}$ almost everywhere, the harmonic extension of φ to the unit disk $\mathbb{D}$ is separated away from zero near the unit circle and $\operatorname{wind}\varphi$ is defined as the winding number of the restriction of the harmonic extension of φ to the circle of radius ρ for ρ sufficiently close to 1 (see [D]). Note also that if $\varphi \in QC$ and φ has constant modulus on $\mathbb{T}$, then the Toeplitz operator T_φ on H^2 is Fredholm and its index $\operatorname{ind} T_\varphi$ equals $-\operatorname{wind}\varphi$ (see [D]). Recall that for $\varphi \in L^\infty$ the Toeplitz operator T_φ on H^2 is defined by

$$T_\varphi f = \mathbb{P}_+ \varphi f, \quad f \in H^2,$$

where $\mathbb{P}_+$ is the orthogonal projection onto H^2.

A similar description holds for functions $\varphi \in L^\infty$ such that $\|H_\varphi\|_{\mathrm{e}} < \|H_\varphi\|$. In this case φ is badly approximable if and only if φ has constant modulus almost everywhere on $\mathbb{T}$, the Toeplitz operator T_φ is Fredholm and $\operatorname{ind} T_\varphi > 0$.

To state the description of badly approximable and very badly approximable matrix functions obtained in [PY1] and [PT2], we need the notion of a thematic matrix function. Recall that a function $F \in H^\infty(\mathbb{M}_{m,n})$ is called *inner* if $F^*(\zeta)F(\zeta) = I_n$ almost everywhere on $\mathbb{T}$ (I_n stands for the identity matrix in

$\mathbb{M}_{n,n})$. F is called *outer* if $FH^2(\mathbb{C}^n)$ is dense in $H^2(\mathbb{C}^m)$. Finally, F is called *co-outer* if the transposed function F^{t} is outer.

An $n \times n$ matrix function V, $n \geq 2$, is called *thematic* if it is unitary-valued and has the form

$$V = \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix},$$

where the matrix functions $\boldsymbol{v} \in H^\infty(\mathbb{C}^n)$ and $\Theta \in H^\infty(\mathbb{M}_{n,n-1})$ are both inner and co-outer. Note that if V is a thematic function, then all minors of V on the first column (i.e., minors of an arbitrary size that involve the first column) belong to H^∞ ([PY1]). If $n = 1$, a thematic function is a constant function whose modulus is equal to 1.

It was shown in [PY1] that a function $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n}) \setminus H^\infty(\mathbb{M}_{m,n})$ is badly approximable if and only if it admits a representation

$$\Phi = W^* \begin{pmatrix} su & 0 \\ 0 & \Psi \end{pmatrix} V^*, \tag{2.1}$$

where $s > 0$, V and W^{t} are thematic functions, u is a scalar *unimodular* function (i.e., $|u(\zeta)| = 1$ for almost all $\zeta \in \mathbb{T}$) in QC with negative winding number, and $\|\Psi\|_{L^\infty} \leq s$. Note that in this case V and W must belong to QC, Ψ must belong to $H^\infty + C$, and $s = \|H_\Phi\|$ (see [PY1]).

A similar result was obtained in [PT2] in the more general case when $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$. Such a matrix function Φ is badly approximable if and only if it admits a representation of the form (2.1) in which $s > 0$, $\|\Psi\|_{L^\infty} \leq s$, V and W^{t} are thematic matrix functions , and u is a unimodular function such that T_u is Fredholm and $\operatorname{ind} T_u > 0$.

Suppose now that $m \leq n$. It was proved in [PY1] that a matrix function $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$ is very badly approximable if and only if Φ admits a representation

$$\Phi = W_0^* \cdots W_{m-1}^* \begin{pmatrix} s_0 u_0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & s_1 u_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{m-1} u_{m-1} & 0 & \cdots & 0 \end{pmatrix} V_{m-1}^* \cdots V_0^* \tag{2.2}$$

for some badly approximable unimodular functions $u_0, \cdots, u_{m-1} \in QC$ and some nonincreasing sequence $\{s_j\}_{0 \leq j \leq m-1}$ of nonnegative numbers;

$$W_j = \begin{pmatrix} I_j & 0 \\ 0 & \breve{W}_j \end{pmatrix}, \quad V_j = \begin{pmatrix} I_j & 0 \\ 0 & \breve{V}_j \end{pmatrix}, \quad 1 \leq j \leq m-1, \tag{2.3}$$

and $W_0^{\mathrm{t}}, \breve{W}_j^{\mathrm{t}}, V_0, \breve{V}_j$ are thematic matrix functions, $1 \leq j \leq m-1$. Moreover, in this case the s_j are the superoptimal singular values of Φ: $s_j = t_j$, $0 \leq j \leq m-1$, and the matrix functions V_j, W_j, $0 \leq j \leq m-1$, must belong to QC.

Consider now factorizations of the form (2.2). Suppose that $\{s_j\}_{0\le j\le m-1}$ is a nonincreasing sequence of nonnegative numbers, the matrix functions $W_0^{\mathrm t}, \breve W_j^{\mathrm t}, V_0, \breve V_j$ (see (2.3)) are thematic, the u_j are unimodular functions such that the Toeplitz operators T_{u_j} are Fredholm and $\operatorname{ind} T_{u_j} > 0$. Such factorizations are called *thematic factorizations*.

It was shown in [PT2] that if $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $\|H_\Phi\|_{\mathrm e}$ is less than the smallest nonzero superoptimal singular value of Φ, then Φ is very badly approximable if and only if it admits a thematic factorization.

The indices k_j of the thematic factorization (2.2) (*thematic indices*) are defined in case $t_j \ne 0$: $k_j \stackrel{\mathrm{def}}{=} \operatorname{ind} T_{u_j}$ (recall that if $u_j \in QC$, then $k_j = -\operatorname{wind} u_j$).

It follows from the results of [PY1] that if $\Phi \in L^\infty(\mathbb{M}_{m,n})$ admits a representation (2.1) in which $s > 0$, V and $W^{\mathrm t}$ are thematic matrix functions, u is a unimodular function such that T_u is Fredholm with $\operatorname{ind} T_u > 0$, and $\|\Psi\|_{L^\infty} \le s$, then Φ is a badly approximable matrix function. If Φ admits a thematic factorization (2.2), then Φ is very badly approximable with superoptimal singular values s_j, $0 \le j \le m-1$ (see [PY1]).

It also follows from the results of [PT2] that if $\|H_\Phi\|_{\mathrm e} < \|H_\Phi\|$, $r \le \min\{m,n\}$ is such that $t_{r-1} > \|H_\Phi\|_{\mathrm e}$ and $t_{r-1} > t_r$, and $F \in \Omega_{r-1}$, then $\Phi - F$ admits a factorization

$$\Phi - F = W_0^* \cdots W_{r-1}^* \begin{pmatrix} t_0u_0 & 0 & \cdots & 0 & 0 \\ 0 & t_1u_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_{r-1}u_{r-1} & 0 \\ 0 & 0 & \cdots & 0 & \Psi \end{pmatrix} V_{r-1}^* \cdots V_0^*, \tag{2.4}$$

in which the V_j and W_j have the form (2.3), the $W_0^{\mathrm t}, \breve W_j^{\mathrm t}, V_0, \breve V_j$ are thematic matrix functions, the u_j are unimodular functions such that T_{u_j} is Fredholm and $\operatorname{ind} T_{u_j} > 0$,

$$\|\Psi\|_{L^\infty} \le t_{r-1} \quad \text{and} \quad \|H_\Psi\| < t_{r-1}. \tag{2.5}$$

Factorizations of the form (2.4) with a nonincreasing sequence $\{t_j\}_{0\le j\le r-1}$ and Ψ satisfying (2.5) are called *partial thematic factorizations*. If $\Phi - F$ admits a partial thematic factorization of the form (2.4), then $t_0, t_1, \cdots, t_{r-1}$ are the largest r superoptimal singular values of Φ, and so they do not depend on the choice of a partial thematic factorization.

The matrix entry Ψ in the partial thematic factorization (2.4) is called the *residual entry* of the partial thematic factorization.

3. **Invariance of residual entries**

The aim of this section is to show that if a matrix function admits a partial thematic factorization of the form (2.4), then the residual entry Ψ in (2.4) is uniquely determined by the function itself modulo constant unitary factors.

Lemma 3.1. *Let Φ be an $m \times n$ matrix of the form*

$$\Phi = W^* \begin{pmatrix} u & 0 \\ 0 & \Psi \end{pmatrix} V^*,$$

where $m, n \geq 2$, $u \in \mathbb{C}$, $\Psi \in \mathbb{M}_{m-1,n-1}$, and

$$V = \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix} \in \mathbb{M}_{n,n}, \quad W = \begin{pmatrix} \boldsymbol{w} & \Xi \end{pmatrix}^t \in \mathbb{M}_{m,m}$$

are unitary matrices such that $\boldsymbol{v} \in \mathbb{M}_{n,1}$ and $\boldsymbol{w} \in \mathbb{M}_{m,1}$. Then

$$\Psi = \Xi^* \Phi \overline{\Theta}.$$

Proof. We have

$$\begin{aligned}
\Xi^* \Phi \overline{\Theta} &= \Xi^* W^* \begin{pmatrix} u & 0 \\ 0 & \Psi \end{pmatrix} V^* \overline{\Theta} \\
&= \Xi^* \begin{pmatrix} \overline{\boldsymbol{w}} & \Xi \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \Psi \end{pmatrix} \begin{pmatrix} \boldsymbol{v}^* \\ \Theta^{\mathrm{t}} \end{pmatrix} \overline{\Theta} \\
&= \begin{pmatrix} 0 & I_{m-1} \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \Psi \end{pmatrix} \begin{pmatrix} 0 \\ I_{n-1} \end{pmatrix} = \Psi. \quad \blacksquare
\end{aligned}$$

Corollary 3.2. *Let Φ be an $m \times n$ matrix of the form*

$$\Phi = W_0^* \cdots W_{r-1}^* \begin{pmatrix} \varphi_0 & 0 & \cdots & 0 & 0 \\ 0 & \varphi_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \varphi_{r-1} & 0 \\ 0 & 0 & \cdots & 0 & \Psi \end{pmatrix} V_{r-1}^* \cdots V_0^*,$$

where $r < \min\{m, n\}$, $\varphi_0, \varphi_1, \cdots, \varphi_{r-1} \in \mathbb{C}$,

$$V_j = \begin{pmatrix} I_j & 0 \\ 0 & \breve{V}_j \end{pmatrix}, \quad W_j = \begin{pmatrix} I_j & 0 \\ 0 & \breve{W}_j \end{pmatrix},$$

are unitary matrices such that

$$\breve{V}_j = \begin{pmatrix} \boldsymbol{v}_j & \overline{\Theta_j} \end{pmatrix}, \quad \breve{W}_j = \begin{pmatrix} \boldsymbol{w}_j & \overline{\Xi_j} \end{pmatrix}^{\mathrm{t}}, \quad 0 \leq j \leq r-1,$$

$\boldsymbol{v}_j \in \mathbb{M}_{n-j,1}$, $\boldsymbol{w}_j \in \mathbb{M}_{m-j,1}$. Then

$$\Psi = \Xi_{r-1}^* \cdots \Xi_1^* \Xi_0^* \Phi \overline{\Theta_0 \Theta_1 \cdots \Theta_{r-1}}.$$

Proof. The result follows immediately from Lemma 3.1 by induction. $\blacksquare$

Theorem 3.3. *Suppose that a matrix function $\Phi \in L^\infty(\mathbb{M}_{m,n})$ admits partial thematic factorizations*

$$\Phi = W_0^* \cdots W_{r-1}^* \begin{pmatrix} t_0 u_0 & 0 & \cdots & 0 & 0 \\ 0 & t_1 u_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_{r-1} u_{r-1} & 0 \\ 0 & 0 & \cdots & 0 & \Psi \end{pmatrix} V_{r-1}^* \cdots V_0^*,$$

and

$$\Phi = (W_0^\heartsuit)^* \cdots (W_{r-1}^\heartsuit)^* \begin{pmatrix} t_0 u_0^\heartsuit & 0 & \cdots & 0 & 0 \\ 0 & t_1 u_1^\heartsuit & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_{r-1} u_{r-1}^\heartsuit & 0 \\ 0 & 0 & \cdots & 0 & \Psi^\heartsuit \end{pmatrix} (V_{r-1}^\heartsuit)^* \cdots (V_0^\heartsuit)^*.$$

Then there exist constant unitary matrices $U_1 \in \mathbb{M}_{n-r,n-r}$ and $U_2 \in \mathbb{M}_{m-r,m-r}$ such that

$$\Psi^\heartsuit = U_2 \Psi U_1.$$

Recall that by the definition of a partial thematic factorization, Ψ must satisfy (2.5), and this is very important.

Proof. Let

$$V_j = \begin{pmatrix} I_j & 0 \\ 0 & \breve{V}_j \end{pmatrix}, \quad W_j = \begin{pmatrix} I_j & 0 \\ 0 & \breve{W}_j \end{pmatrix},$$

and

$$V_j^\heartsuit = \begin{pmatrix} I_j & 0 \\ 0 & \breve{V}_j^\heartsuit \end{pmatrix}, \quad W_j^\heartsuit = \begin{pmatrix} I_j & 0 \\ 0 & \breve{W}_j^\heartsuit \end{pmatrix},$$

where

$$\breve{V}_j = \begin{pmatrix} \boldsymbol{v}_j & \overline{\Theta_j} \end{pmatrix}, \quad \breve{W}_j = \begin{pmatrix} \boldsymbol{w}_j & \overline{\Xi_j} \end{pmatrix}^{\mathrm{t}}, \quad 0 \le j \le r-1,$$

$$\breve{V}_j^\heartsuit = \begin{pmatrix} \boldsymbol{v}_j^\heartsuit & \overline{\Theta_j^\heartsuit} \end{pmatrix}, \quad \breve{W}_j^\heartsuit = \begin{pmatrix} \boldsymbol{w}_j^\heartsuit & \overline{\Xi_j^\heartsuit} \end{pmatrix}^{\mathrm{t}}, \quad 0 \le j \le r-1.$$

Here $\breve{V}_0 \stackrel{\mathrm{def}}{=} V_0$, $\breve{W}_0 \stackrel{\mathrm{def}}{=} V_0$, $\breve{V}_0^\heartsuit \stackrel{\mathrm{def}}{=} V_0^\heartsuit$, and $\breve{W}_0^\heartsuit \stackrel{\mathrm{def}}{=} W_0^\heartsuit$.

We need the following lemma.

Lemma 3.4.

$$\Theta_0 \Theta_1 \cdots \Theta_{r-1} H^2(\mathbb{C}^{n-r}) = \Theta_0^\heartsuit \Theta_1^\heartsuit \cdots \Theta_{r-1}^\heartsuit H^2(\mathbb{C}^{n-r}) \tag{3.1}$$

and

$$\Xi_0 \Xi_1 \cdots \Xi_{r-1} H^2(\mathbb{C}^{m-r}) = \Xi_0^\heartsuit \Xi_1^\heartsuit \cdots \Xi_{r-1}^\heartsuit H^2(\mathbb{C}^{m-r}). \tag{3.2}$$

Let us first complete the proof of Theorem 3.3. Consider the inner matrix functions

$$\Theta = \Theta_0\Theta_1\cdots\Theta_{r-1}, \quad \Theta^\heartsuit = \Theta_0^\heartsuit\Theta_1^\heartsuit\cdots\Theta_{r-1}^\heartsuit$$

and

$$\Xi = \Xi_0\Xi_1\cdots\Xi_{r-1}, \quad \Xi^\heartsuit = \Xi_0^\heartsuit\Xi_1^\heartsuit\cdots\Xi_{r-1}^\heartsuit.$$

By Lemma 3.4, $\Theta H^2(\mathbb{C}^{n-r}) = \Theta^\heartsuit H^2(\mathbb{C}^{n-r})$. It is well known that in this case there exists a constant unitary matrix $Q_1 \in \mathbb{M}_{n-r,n-r}$ such that $\Theta^\heartsuit = \Theta Q_1$ (Θ and $\Theta^\heartsuit$ determine the same invariant subspace under multiplication by z, see e.g., [N]). Similarly, there exists a constant unitary matrix $Q_2 \in \mathbb{M}_{m-r,m-r}$ such that $\Xi^\heartsuit = \Xi Q_2$.

By Corollary 3.2,

$$\Psi = \Xi^*\Phi\overline{\Theta}, \quad \Psi^\heartsuit = (\Xi^\heartsuit)^*\Phi\overline{\Theta^\heartsuit}.$$

Hence,

$$\Psi^\heartsuit = Q_2^*\Xi^*\Phi\overline{\Theta Q_1} = Q_2^*\Psi\overline{Q_1}. \quad \blacksquare$$

Proof of Lemma 3.4. It is sufficient to prove (3.1). Indeed, (3.2) follows from (3.1) applied to Φ^{t}.

It is easy to see that without loss of generality we may assume that $\|\Psi\|_{L^\infty} < t_{r-1}$. Indeed, we can subtract from Φ a matrix function in Ω_{r-1}, and it follows from Lemma 1.5 of [PY1] that the resulting function admits a partial thematic factorization with the same unitary-valued function V_j and W_j, $0 \le j \le r-1$, and residual entry whose L^∞ norm is less that t_{r-1}. It is also easy to see that if $\|\Psi\|_{L^\infty} < t_{r-1}$, then $\|\Psi^\heartsuit\|_{L^\infty}$ must also be less than t_{r-1}.

Consider the subspace $\mathcal{L}$ of $H^2(\mathbb{C}^n)$ defined by

$$\mathcal{L} = \left\{ f \in H^2(\mathbb{C}^n) : \ V_{r-1}^{\mathrm{t}}\cdots V_1^{\mathrm{t}}V_0^{\mathrm{t}}f = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ * \\ \vdots \\ * \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} r \\ \vphantom{\begin{matrix}*\\ \vdots\\ *\end{matrix}} \end{matrix} \right\},$$

i.e., $\mathcal{L}$ consists of vector functions $f \in H^2(\mathbb{C}^n)$ such that the first r components of the vector function $V_{r-1}^{\mathrm{t}}\cdots V_1^{\mathrm{t}}V_0^{\mathrm{t}}f$ are zero.

We define the real function ρ on $\mathbb{R}$ by

$$\rho(x) = \begin{cases} x, & x \ge t_{r-1}^2 \\ 0, & x < t_{r-1}^2 \end{cases}$$

and consider the operator $M : H^2(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ of multiplication by the matrix function $\rho(\Phi^{\mathrm{t}}\overline{\Phi})$:

$$Mf = \rho(\Phi^{\mathrm{t}}\overline{\Phi})f, \quad f \in H^2(\mathbb{C}^n).$$

Let us show that

$$\mathcal{L} = \operatorname{Ker} M. \tag{3.3}$$

We have

$$\Phi^{\mathrm{t}}\overline{\Phi} = \overline{V_0 V_1 \cdots V_{r-1}} \begin{pmatrix} t_0^2 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & t_{r-1}^2 & 0 \\ 0 & \cdots & 0 & \Psi^{\mathrm{t}}\overline{\Psi} \end{pmatrix} V_{r-1}^{\mathrm{t}} \cdots V_1^{\mathrm{t}} V_0^{\mathrm{t}},$$

and since $\|\Psi^{\mathrm{t}}\overline{\Psi}\|_{L^\infty} < t_{r-1}^2$, it follows that

$$\rho(\Phi^{\mathrm{t}}\overline{\Phi}) = \overline{V_0 V_1 \cdots V_{r-1}} \begin{pmatrix} t_0^2 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & t_{r-1}^2 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} V_{r-1}^{\mathrm{t}} \cdots V_1^{\mathrm{t}} V_0^{\mathrm{t}}.$$

Since all matrix functions V_j are unitary-valued, this implies (3.3).

Thus the subspace $\mathcal{L}$ is uniquely determined by the function Φ and does not depend on the choice of a partial thematic factorization. It is easy to see that to complete the proof of Lemma 3.4, it is sufficient to prove the following lemma.

Lemma 3.5.

$$\mathcal{L} = \Theta_0 \Theta_1 \cdots \Theta_{r-1} H^2(\mathbb{C}^{n-r}). \tag{3.4}$$

Proof. We show by induction on r that (3.4) holds even without the assumption that $\|\Psi\|_{L^\infty} < t_{r-1}$ (note that this assumption is very important in the proof of (3.3)).

Suppose that $r = 1$. Then

$$\mathcal{L} = \left\{ f \in H^2(\mathbb{C}^n) : \ V_0^{\mathrm{t}} f = \begin{pmatrix} 0 \\ * \\ \vdots \\ * \end{pmatrix} \right\}.$$

Obviously, if $f \in \Theta_0 H^2(\mathbb{C}^{n-1})$, then $f \in \mathcal{L}$. Suppose now that $f \in \mathcal{L}$. We have

$$V_0^{\mathrm{t}} f = \begin{pmatrix} 0 \\ g \end{pmatrix}, \quad g \in L^2(\mathbb{C}^{n-1}).$$

Then

$$f = \overline{V_0} \begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} \overline{\boldsymbol{v}_0} & \Theta_0 \end{pmatrix} \begin{pmatrix} 0 \\ g \end{pmatrix} = \Theta_0 g.$$

Let us show that $g \in H^2(\mathbb{C}^{n-1})$. It suffices to prove that $g^{\mathrm{t}}\gamma \in H^2$ for any constant vector $\gamma \in \mathbb{C}^{n-1}$. Since Θ_0^{t} is outer, there exists a sequence $\{\varphi_n\}_{n\ge 0}$ of functions in $H^2(\mathbb{C}^n)$ such that

$$\lim_{n\to\infty} \Theta_0^{\mathrm{t}}\varphi_n \to \gamma \quad \text{in} \quad H^2(\mathbb{C}^{n-1}).$$

We have

$$f^{\mathrm{t}}\varphi_n = g^{\mathrm{t}}\Theta_0^{\mathrm{t}}\varphi_n \to g^{\mathrm{t}}\gamma \quad \text{in} \quad H^1,$$

and so $g^{\mathrm{t}}\gamma \in H^2$ which proves the result for $r = 1$.

Suppose now that $r \ge 2$. By the induction hypothesis

$$\mathcal{L} = \left\{ \Theta_0 \cdots \Theta_{r-2} g : \; g \in H^2(\mathbb{C}^{n-r+1}), \; V_{r-1}^{\mathrm{t}} \cdots V_0^{\mathrm{t}} \Theta_0 \cdots \Theta_{r-2} g = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ * \\ \vdots \\ * \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} r \\ \vphantom{\begin{matrix}*\\ \vdots\\ *\end{matrix}} \end{matrix} \right\}.$$

It follows from the definition of thematic matrix functions that

$$V_{r-2}^{\mathrm{t}} \cdots V_0^{\mathrm{t}} \Theta_0 \cdots \Theta_{r-2} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} r-1 \\ \left.\vphantom{\begin{matrix}1\\ \vdots\\ 0\end{matrix}}\right\} n-r+1 \end{matrix} = \begin{pmatrix} 0 \\ I_{n-r+1} \end{pmatrix}.$$

Hence,

$$\mathcal{L} = \left\{ \Theta_0 \cdots \Theta_{r-2} g : \; g \in H^2(\mathbb{C}^{n-r+1}), \; \begin{pmatrix} \boldsymbol{v}_{r-1}^{\mathrm{t}} \\ \Theta_{r-1}^* \end{pmatrix} g = \begin{pmatrix} 0 \\ * \\ \vdots \\ * \end{pmatrix} \right\}.$$

Since the result has already been proved for $r = 1$,

$$\mathcal{L} = \left\{ \Theta_0 \cdots \Theta_{r-2} g : \; g \in \Theta_{r-1} H^2(\mathbb{C}^{n-r}) \right\} = \Theta_0 \cdots \Theta_{r-1} H^2(\mathbb{C}^{n-r}). \quad \blacksquare$$

4. **Monotone thematic factorizations and invariance of indices**

In this section we study the problem of the invariance of indices of thematic factorizations of very badly approximable matrix functions. In [PY1] it was shown that the indices of a thematic factorization are not determined uniquely by the

matrix function but may depend on the choice of a thematic factorization. For example, the matrix function $\Phi = \begin{pmatrix} \bar{z}^2 & 0 \\ 0 & \bar{z}^6 \end{pmatrix}$ admits the following thematic factorizations

$$
\begin{aligned}
\Phi &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{z}^2 & 0 \\ 0 & \bar{z}^6 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{z^5}{\sqrt{2}} \\ \frac{\bar{z}^5}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \bar{z} & 0 \\ 0 & \bar{z}^7 \end{pmatrix} \begin{pmatrix} \frac{\bar{z}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{z}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{z}^6 & 0 \\ 0 & \bar{z}^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}
$$

The superoptimal singular values of Φ are $t_0 = t_1 = 1$. The indices of the first factorization are 2, 6, the indices of the second are 1, 7, and the indices of the third are 6, 2. Note that for all above factorizations the sum of the indices is 8.

In [PY2] it was shown (in the case of $H^\infty + C$ functions) that the sum of thematic indices that correspond to all superoptimal singular values equal to a positive specific value does not depend on the choice of a thematic factorization. In other words, for each positive superoptimal singular value t the numbers

$$\nu_t \stackrel{\text{def}}{=} \sum_{\{j:t_j=t\}} k_j$$

do not depend on the choice of a thematic factorization. The same result was obtained in [PT2] in the case when $\|H_\Phi\|_{\mathrm{e}}$ is less than the smallest nonzero superoptimal singular value. Note that it also follows from the results of [PY2] and [PT2] that the same invariance property holds for partial thematic factorizations.

A natural question arises of whether we can distribute arbitrarily the numbers ν_t between the indices k_j with $t_j = t$ by choosing an appropriate thematic factorization (recall that the k_j must be positive integers).

In this section we show that the answer to this question is negative.

Definition. A (partial) thematic factorization is called *monotone* if for any positive superoptimal singular value t the thematic indices $k_r, k_{r+1}, \cdots, k_s$ that correspond to all superoptimal singular values equal to t satisfy

$$k_r \geq k_{r+1} \geq \cdots \geq k_s. \tag{4.1}$$

Here $t_r, t_{r+1}, \cdots, t_s$ are the superoptimal singular values equal to t.

We prove in this section that if $\|H_\Phi\|_{\mathrm{e}}$ is less than the smallest nonzero superoptimal singular value of Φ, then $\Phi - \mathcal{A}\Phi$ possesses a monotone thematic factorization. We also show that the indices of a monotone thematic factorization are uniquely

determined by the function Φ itself and do not depend on the choice of a thematic factorization. In particular this is the case if $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$. The same results also hold for partial thematic factorizations.

In the above example only the third thematic factorization is monotone. It will follow from the results of this section that the thematic indices of any monotone thematic factorization must be equal to 6, 2. In particular, there are no thematic factorizations with indices 7, 1. Note that it is important that the indices in (4.1) are arranged in the *nonincreasing* order. The above example shows that the first two thematic factorizations have different thematic indices 2, 6 and 1, 7 that are arranged in the increasing order.

Theorem 4.1. *Suppose that $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $r \le \min\{m,n\}$ is a positive integer such that the superoptimal singular values of Φ satisfy*

$$t_{r-1} > t_r, \quad t_{r-1} > \|H_\Phi\|_{\mathrm{e}}.$$

If Φ admits a partial thematic factorization of the form (2.4)*, then Φ admits a monotone partial thematic factorization of the form* (2.4)*.*

Proof. Clearly, $\|H_{z^j\Phi}\| = \operatorname{dist}_{L^\infty}(\Phi, \bar{z}^j H^\infty(\mathbb{M}_{m,n}))$, and it is easy to see that

$$\lim_{j\to\infty} \|H_{z^j\Phi}\| = \operatorname{dist}_{L^\infty}\big(\Phi, (H^\infty + C)(\mathbb{M}_{m,n})\big) = \|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|.$$

Put

$$\iota(H_\Phi) \stackrel{\text{def}}{=} \min\{j \ge 0: \ \|H_{z^j\Phi}\| < \|H_\Phi\|\}.$$

Obviously, $\iota(H_\Phi)$ depends only on the Hankel operator H_Φ and does not depend on the choice of its symbol.

We need three lemmas.

Lemma 4.2. *Let Φ be a matrix function in $L^\infty(\mathbb{M}_{m,n})$ such that $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$. Suppose that*

$$\Phi = W^* \begin{pmatrix} tu & 0 \\ 0 & \Upsilon \end{pmatrix} V^*, \tag{4.2}$$

where V and W^{t} are thematic matrix functions of sizes $n \times n$ and $m \times m$, $t > 0$, $\|\Upsilon\|_{L^\infty} \le t$, and u is a unimodular function such that T_u is Fredholm. Then $\operatorname{ind} T_u \le \iota(H_\Phi)$*.*

Lemma 4.3. *Let Φ be a badly approximable matrix function in $L^\infty(\mathbb{M}_{m,n})$ such that $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$. Then Φ admits a representation* (4.2) *with thematic matrix functions V and W^{t}, $t = t_0 = \|H_\Phi\|$, and a unimodular function u such that T_u is Fredholm and*

$$\operatorname{ind} T_u = \iota(H_\Phi).$$

Lemma 4.4. *Let $\Phi \in L^\infty(\mathbb{M}_{m,n})$ be a matrix function of the form*

$$\Phi = W^* \begin{pmatrix} u & 0 \\ 0 & \Upsilon \end{pmatrix} V^*,$$

where V and W^{t} are thematic matrix functions of sizes $n \times n$ and $m \times m$, u is a unimodular function such that T_u is Fredholm, $\operatorname{ind} T_u = 0$, $\|H_\Upsilon\| \le 1$, and $\|H_\Upsilon\|_{\mathrm{e}} < 1$. If $\|H_\Phi\| < 1$, then $\|H_\Upsilon\| < 1$.

Let us first complete the proof of Theorem 4.1. We argue by induction on r. For $r = 1$ the result is trivial. Suppose now that $r > 1$. By Lemma 4.3, Φ admits a representation

$$\Phi = W^* \begin{pmatrix} t_0 u_0 & 0 \\ 0 & \Upsilon \end{pmatrix} V^*,$$

where V and W^{t} are thematic functions, $\|\Upsilon\|_{L^\infty} \le t_0$, and u_0 is a unimodular function such that T_{u_0} is Fredholm and $\operatorname{ind} T_{u_0} = \iota(H_\Phi)$. By Theorem 6.3 of [PT2],

$$\|H_\Upsilon\|_{\mathrm{e}} \le \|H_\Phi\|_{\mathrm{e}}. \tag{4.3}$$

It follows from the results of §4 and §6 of [PT2] that Υ admits a partial thematic factorization of the form

$$\Upsilon = W_1^* \cdots W_{r-1}^* \begin{pmatrix} t_1 u_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & t_{r-1} u_{r-1} & 0 \\ 0 & \cdots & 0 & \Psi \end{pmatrix} V_{r-1}^* \cdots V_1^*.$$

By the induction hypothesis we may assume that this partial thematic factorization is monotone. Clearly, $t_1 = \|\Upsilon\|_{L^\infty}$. If $t_1 < t_0$, then it is easy to see that the above factorization of Υ leads to a monotone partial thematic factorization of Φ.

Suppose now that $t_1 = t_0$. To prove that the above factorization of Υ leads to a monotone partial thematic factorization of Φ, we have to establish the inequality $\operatorname{ind} T_{u_0} \ge \operatorname{ind} T_{u_1}$. By Lemma 4.2, $\iota(H_\Upsilon) \ge \operatorname{ind} T_{u_1}$, and it suffices to prove the inequality

$$\iota(H_\Phi) = \operatorname{ind} T_{u_0} \ge \iota(H_\Upsilon).$$

Put $\iota \stackrel{\mathrm{def}}{=} \iota(H_\Phi)$. We have

$$z^\iota \Phi = W^* \begin{pmatrix} t_0 z^\iota u_0 & 0 \\ 0 & z^\iota \Upsilon \end{pmatrix} V^*.$$

Clearly, $\operatorname{ind} T_{z^\iota u_0} = 0$. By the definition of ι, $\|H_{z^\iota \Phi}\| < \|H_\Phi\| = t_0$. It is easy to see that

$$\|H_{z^\iota \Upsilon}\|_{\mathrm{e}} = \|H_\Upsilon\|_{\mathrm{e}} < t_0$$

by (4.3). It follows from Lemma 4.4 that $\|H_{z^\iota \Upsilon}\| < t_0$ which means that $\iota(H_\Upsilon) \le \iota$. ■

Proof of Lemma 4.2. Let $k = \operatorname{ind} T_u$. Clearly, it is sufficient to consider the case $k > 0$. Then Φ is badly approximable and $\|H_\Phi\| = t$ (see §2). We have

$$z^{k-1}\Phi = W^* \begin{pmatrix} tz^{k-1}u & 0 \\ 0 & z^{k-1}\Upsilon \end{pmatrix} V^*.$$

Then $\operatorname{wind}(z^{k-1}u) = -1$, and so $z^{k-1}\Phi$ is badly approximable and $\|\Phi\|_{L^\infty} = t$ (see §2). Hence,

$$\|H_{z^{k-1}\Phi}\| = \|z^{k-1}\Phi\|_{L^\infty} = \|\Phi\|_{L^\infty} = t = \|H_\Phi\|,$$

and so $\iota(H_\Phi) \ge k$. ■

Proof of Lemma 4.3. Put $\iota \stackrel{\text{def}}{=} \iota(H_\Phi)$. Then

$$\|H_{z^{\iota-1}\Phi}\| = \|H_\Phi\| = \|\Phi\|_{L^\infty} = \|z^{\iota-1}\Phi\|_{L^\infty},$$

and so $z^{\iota-1}\Phi$ is badly approximable. Clearly,

$$\|H_{z^{\iota-1}\Phi}\|_{\mathrm{e}} = \|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\| = \|H_{z^{\iota-1}\Phi}\|.$$

Hence, (see §2) $z^{\iota-1}\Phi$ admits a representation

$$z^{\iota-1}\Phi = W^* \begin{pmatrix} t\omega & 0 \\ 0 & \Omega \end{pmatrix} V^*,$$

where $t = \|H_\Phi\|$, ω is a unimodular function such that $\operatorname{ind} T_\omega > 0$, V and W^{t} are thematic functions and $\|\Omega\|_{L^\infty} \le t$. Therefore

$$\Phi = W^* \begin{pmatrix} t\bar{z}^{\iota-1}\omega & 0 \\ 0 & \bar{z}^{\iota-1}\Omega \end{pmatrix} V^*.$$

Let $u = \bar{z}^{\iota-1}\omega$. Clearly, $\operatorname{ind} T_u \ge \iota$. Finally, by Lemma 4.2, $\operatorname{ind} T_u = \iota$. ■

Proof of Lemma 4.4. The proof is based on the argument given in the proof of Lemma 1.2 of [PY2]. Let

$$V = \begin{pmatrix} \boldsymbol{v} & \overline{\Theta} \end{pmatrix}, \quad W^{\mathrm{t}} = \begin{pmatrix} \boldsymbol{w} & \overline{\Xi} \end{pmatrix}.$$

By Theorem 5.1 of [PT2], there exist $A \in H^\infty(\mathbb{M}_{n-1,n})$ and $B \in H^\infty(\mathbb{M}_{m-1,m})$ such that $A\Theta = I_{n-1}$ and $B\Xi = I_{m-1}$. Without loss of generality we may assume that $\|\Upsilon\|_{L^\infty} \le 1$.

Suppose that $\|H_\Upsilon\| = 1$. Since $\|H_\Upsilon\|_{\mathrm{e}} < 1$, there exists a nonzero function $g \in H^2(\mathbb{C}^{n-1})$ such that $\|H_\Upsilon g\|_2 = \|g\|_2$. Then $\Upsilon g \in H^2_-(\mathbb{C}^{m-1})$ and $\|\Upsilon(\zeta)g(\zeta)\|_{\mathbb{C}^{m-1}} = \|g(\zeta)\|_{\mathbb{C}^{n-1}}$ for almost all $\zeta \in \mathbb{T}$.

Let

$$f = A^{\mathrm{t}}g + \boldsymbol{v}q,$$

where q is a scalar function in H^2. We want to find such a q that $\|H_\Phi f\|_2 = \|f\|_2$. Note that f is a nonzero function since

$$V^* f = \begin{pmatrix} \boldsymbol{v}^* \\ \Theta^{\mathrm{t}} \end{pmatrix} (A^{\mathrm{t}}g + \boldsymbol{v}q) = \begin{pmatrix} \boldsymbol{v}^* A^{\mathrm{t}} g + q \\ g \end{pmatrix}$$

and $g \neq 0$.

We have

$$\begin{aligned}
\Phi f &= W^* \begin{pmatrix} u & 0 \\ 0 & \Upsilon \end{pmatrix} \begin{pmatrix} \boldsymbol{v}^* A^{\rm t} g + q \\ g \end{pmatrix} \\
&= \begin{pmatrix} \overline{\boldsymbol{w}} & \Xi \end{pmatrix} \begin{pmatrix} u\boldsymbol{v}^* A^{\rm t} g + uq \\ \Upsilon g \end{pmatrix} \\
&= \overline{\boldsymbol{w}}(u\boldsymbol{v}^* A^{\rm t} g + uq) + \Xi\Upsilon g.
\end{aligned}$$

Since the matrix functions W^* and V^* are unitary-valued and $\|\Upsilon(\zeta)g(\zeta)\|_{\mathbb{C}^{m-1}} = \|g(\zeta)\|_{\mathbb{C}^{n-1}}$, it follows that $\|\Phi(\zeta)f(\zeta)\|_{\mathbb{C}^m} = \|f(\zeta)\|_{\mathbb{C}^n}$. It remains to choose q so that $\Phi f \in H^2_-(\mathbb{C}^m)$.

Since W^* is a unitary-valued matrix function, we have

$$I_m = \begin{pmatrix} \overline{\boldsymbol{w}} & \Xi \end{pmatrix} \begin{pmatrix} \boldsymbol{w}^{\rm t} \\ \Xi^* \end{pmatrix} = \overline{\boldsymbol{w}}\boldsymbol{w}^{\rm t} + \Xi\Xi^*.$$

Hence,

$$\Xi = \Xi(B\Xi)^* = \Xi\Xi^* B^* = (I_m - \overline{\boldsymbol{w}}\boldsymbol{w}^{\rm t})B^*.$$

It follows that

$$\begin{aligned}
\Phi f &= \overline{\boldsymbol{w}}(u\boldsymbol{v}^* A^{\rm t} g + uq) + (I_m - \overline{\boldsymbol{w}}\boldsymbol{w}^{\rm t})B^*\Upsilon g \\
&= \overline{\boldsymbol{w}}(u\boldsymbol{v}^* A^{\rm t} g + uq - \boldsymbol{w}^{\rm t} B^*\Upsilon g) + B^*\Upsilon g.
\end{aligned}$$

Clearly, $B^*\Upsilon g \in H^2_-(\mathbb{C}^m)$, and so it suffices to find $q \in H^2$ such that

$$u\boldsymbol{v}^* A^{\rm t} g + uq - \boldsymbol{w}^{\rm t} B^*\Upsilon g \in H^2_-$$

which is equivalent to the condition

$$T_u q = \mathbb{P}_+(\boldsymbol{w}^{\rm t} B^*\Upsilon g - u\boldsymbol{v}^* A^{\rm t} g).$$

The existence of such a q follows from the well-known fact that the Toeplitz operator T_u is invertible; indeed, it is Fredholm and $\operatorname{ind} T_u = 0$ (see e.g., [D] or [N]). ■

Corollary 4.5. *Let Φ be a very badly approximable matrix function in $L^\infty(\mathbb{M}_{m,n})$ such that $\|H_\Phi\|_{\rm e}$ is less than the smallest nonzero superoptimal singular value of Φ. Then Φ admits a monotone thematic factorization.*

Corollary 4.6. *Let Φ be a very badly approximable matrix function in $(H^\infty + C)(\mathbb{M}_{m,n})$. Then Φ admits a monotone thematic factorization.*

We are going to prove now that the indices of a monotone thematic factorization are uniquely determined by the function itself. We need the following lemma.

Lemma 4.7. *Suppose that a matrix function $\Phi \in L^\infty(\mathbb{M}_{m,n})$ admits a factorization of the form*

$$\Phi = W_0^* \cdots W_{r-1}^* \begin{pmatrix} tu_0 & 0 & \cdots & 0 & 0 \\ 0 & tu_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & tu_{r-1} & 0 \\ 0 & 0 & \cdots & 0 & \Psi \end{pmatrix} V_{r-1}^* \cdots V_0^*,$$

where the V_j and W_j are of the form (2.3), $\|H_\Psi\| < t$ *and the u_j are unimodular functions such that T_{u_j} is Fredholm and* $\operatorname{ind} T_{u_j} \le 0$. *If $\|H_\Phi\|_{\mathrm{e}} < t$, then $\|H_\Phi\| < t$.*

Proof. We argue by induction on r. Let $r = 1$. We have

$$\Phi = W^* \begin{pmatrix} tu & 0 \\ 0 & \Psi \end{pmatrix} V^*,$$

where V and W^{t} are thematic matrix functions, u is a unimodular function such that T_u is Fredholm, $\operatorname{ind} T_u \le 0$, and $\|H_\Psi\| < t$. It follows from Lemma 1.5 of [PY1] that we may subtract from Ψ a best analytic approximation without changing H_Φ, and so we may assume that $\|\Psi\|_{L^\infty} < t$. Without loss of generality we may also assume that $t = 1$.

Suppose that $\|H_\Phi\| = 1$. Since $\|H_\Phi\|_{\mathrm{e}} < 1$, there exists a nonzero function $f \in H^2(\mathbb{C}^n)$ such that $\|H_\Phi f\|_2 = \|f\|_2$. Then $\|\Phi f\|_2 = \|f\|_2$ and since $\|\Psi\|_{L^\infty} < 1$, it follows that $V^* f$ has the form

$$V^* f = \begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.4}$$

Let $\boldsymbol{v}$ be the first column of V. Equality (4.4) means that for almost all $\zeta \in \mathbb{T}$ the remaining columns of $V(\zeta)$ are orthogonal to $f(\zeta)$ in $\mathbb{C}^n$. Since V is unitary-valued, it follows that $f = \xi \boldsymbol{v}$ for a scalar function $\xi \in L^2$. Using the fact that $\boldsymbol{v}$ is co-outer, we can find a sequence of $n \times 1$ functions φ_j in H^2 such that $\lim\limits_{j\to\infty} \|\varphi_j^{\mathrm{t}} \boldsymbol{v} - 1\|_2 = 0$. Hence, ξ is the limit in L^1 of the sequence $\varphi_j^{\mathrm{t}} f$, and so $\xi \in H^2$. Note that $\|f\|_{H^2(\mathbb{C}^n)} = \|\xi\|_{H^2}$.

We have

$$\Phi f = W^* \begin{pmatrix} u\xi \\ 0 \\ \vdots \\ 0 \end{pmatrix} = u\xi \overline{\boldsymbol{w}},$$

where $\boldsymbol{w}$ is the first column of W^{t}. Since f is a maximizing vector of H_Φ, we have $u\xi\overline{\boldsymbol{w}} \in H^2_-(\mathbb{C}^n)$. Again, using the fact that $\boldsymbol{w}$ is co-outer, we find that $u\xi \in H^2_-$,

i.e., $\xi \in \operatorname{Ker} T_u$. However, T_u has trivial kernel since $\operatorname{ind} T_u \le 0$. We have got a contradiction.

Suppose now that $r > 1$. Again, we may assume that $\|\Psi\|_{L^\infty} < t$. Let d be a negative integer such that $d < \operatorname{ind} T_{u_j}$, $0 \le j \le r-1$. Then

$$z^d\Phi = W_0^* \cdots W_{r-1}^* \begin{pmatrix} tz^d u_0 & 0 & \cdots & 0 & 0 \\ 0 & tz^d u_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & tz^d u_{r-1} & 0 \\ 0 & 0 & \cdots & 0 & z^d\Psi \end{pmatrix} V_{r-1}^* \cdots V_0^*$$

is a partial thematic factorization of $z^d\Phi$. Put

$$\Upsilon = W_1^* \cdots W_{r-1}^* \begin{pmatrix} tu_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & tu_{r-1} & 0 \\ 0 & 0 & \cdots & 0 & \Psi \end{pmatrix} V_{r-1}^* \cdots V_1^*$$

Since obviously, $\|H_{z^d\Upsilon}\|_{\mathrm{e}} = \|H_\Upsilon\|_{\mathrm{e}}$ for any $d \in \mathbb{Z}$, it follows from Theorem 6.3 of [PT2] that $\|H_{z^d\Upsilon}\|_{\mathrm{e}} < t$, and so by the induction hypotheses, $\|H_\Upsilon\| < t$. We have

$$\Phi = W_0^* \begin{pmatrix} tu & 0 \\ 0 & \Upsilon \end{pmatrix} V_0^*.$$

The result follows now from the case $r = 1$ which has already been established. ■

Theorem 4.8. *Let Φ be a badly approximable function in $L^\infty(\mathbb{M}_{m,n})$ such that $\|H_\Phi\|_{\mathrm{e}} < \|H_\Phi\|$ and let r be the number of superoptimal singular values of Φ equal to $t_0 = \|H_\Phi\|$. Consider a monotone partial thematic factorization of Φ with indices*

$$k_0 \ge \cdots \ge k_{r-1} \tag{4.5}$$

corresponding to the superoptimal singular values equal to t_0. Let $\kappa \ge 0$. Then

$$\dim\{f \in H^2(\mathbb{C}^n) : \|H_{z^\kappa\Phi} f\|_2 = t_0\|f\|_2\} = \sum_{\{j\in[0,r-1]:k_j>\kappa\}} k_j - \kappa. \tag{4.6}$$

Proof. Let

$$\Phi = \begin{pmatrix} t_0u_0 & 0 & \cdots & 0 & 0 \\ 0 & t_0u_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_0u_{r-1} & 0 \\ 0 & 0 & \cdots & 0 & \Psi \end{pmatrix}$$

be a partial thematic factorization of Φ with indices satisfying (4.5). If $\kappa \ge k_0$, then (4.6) holds by Lemma 4.7. Suppose now that $\kappa < k_0$. Let

$$q = \max\{j \in [0, r-1] : \ k_j > \kappa\}.$$

Clearly, the function $z^\kappa\Phi$ admits the following representation

$$z^\kappa\Phi = W_0^*\cdots W_q^* \begin{pmatrix} t_0 z^\kappa u_0 & 0 & \cdots & 0 & 0 \\ 0 & t_0 z^\kappa u_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_0 z^\kappa u_q & 0 \\ 0 & 0 & \cdots & 0 & \Upsilon \end{pmatrix} V_q^*\cdots V_0^*.$$

where Υ is a matrix function satisfying the hypotheses of Lemma 4.7. By Lemma 4.7, $\|H_\Upsilon\| < t_0$. Let $R \in H^\infty$ be a matrix function such that $\|\Upsilon - R\|_{L^\infty} < t_0$. It is easy to show by induction on q that if we perturb Υ by a bounded analytic matrix function, $z^\kappa\Phi$ also changes by an analytic matrix function (this is the trivial part of Lemma 1.5 of [PY1]). In particular, we can find a matrix function $G \in H^\infty$ such that

$$z^\kappa\Phi - G = W_0^*\cdots W_q^* \begin{pmatrix} t_0 z^\kappa u_0 & 0 & \cdots & 0 & 0 \\ 0 & t_0 z^\kappa u_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_0 z^\kappa u_q & 0 \\ 0 & 0 & \cdots & 0 & \Upsilon - R \end{pmatrix} V_q^*\cdots V_0^*.$$

By Theorem 9.3 of [PT2],

$$\dim\{f \in H^2(\mathbb{C}^n) : \ \|H_{z^\kappa\Phi - G} f\|_2 = t_0\|f\|_2\} = \sum_{\{j\in[0,r-1]:k_j>\kappa\}} k_j - \kappa$$

(this equality was stated in [PT2] for thematic factorizations but the same proof also works for partial thematic factorizations). Equality (4.6) follows now from the obvious fact that $H_{z^\kappa\Phi - G} = H_{z^\kappa\Phi}$. ■

We can now deduce from (4.6) the following result.

Theorem 4.9. *Suppose that $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $q \le \min\{m,n\}$ is a positive integer such that the superoptimal singular values of Φ satisfy*

$$t_{q-1} > t_q, \quad t_{q-1} > \|H_\Phi\|_{\mathrm{e}}$$

and Φ admits a monotone partial thematic factorization

$$\Phi = W_0^*\cdots W_{q-1}^* \begin{pmatrix} t_0 u_0 & 0 & \cdots & 0 & 0 \\ 0 & t_1 u_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_{q-1} u_{q-1} & 0 \\ 0 & 0 & \cdots & 0 & \Upsilon \end{pmatrix} V_{q-1}^*\cdots V_0^*.$$

Then the indices of this factorization are uniquely determined by the function Φ itself.

Proof. Let r be the number of superoptimal singular values equal to $\|H_\Phi\|$. Then Φ admits the following partial thematic factorization

$$\Phi = W_0^* \cdots W_{r-1}^* \begin{pmatrix} t_0 u_0 & 0 & \cdots & 0 & 0 \\ 0 & t_1 u_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_{r-1} u_{r-1} & 0 \\ 0 & 0 & \cdots & 0 & \Psi \end{pmatrix} V_{r-1}^* \cdots V_0^*,$$

where

$$\Psi = W_r^* \cdots W_{q-1}^* \begin{pmatrix} t_r u_r & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & t_{q-1} u_{q-1} & 0 \\ 0 & \cdots & 0 & \Upsilon \end{pmatrix} V_{q-1}^* \cdots V_r^*.$$

By Theorem 3.3, Ψ is determined uniquely by Φ modulo constant unitary factors. Hence, it is sufficient to show that the indices $k_0, \cdots, k_{r-1}$ are uniquely determined by Φ.

It follows easily from (4.6) that

$$k_0 = \min\left\{\kappa : \ \dim\{f \in H^2(\mathbb{C}^n) : \ \|H_{z^\kappa \Phi} f\|_2 = t_0 \|f\|_2\} = 0\right\}.$$

Let now d be the number of indices among $k_0, \cdots, k_{r-1}$ that are to equal to k_0. It follows easily from (4.6) that

$$d = \dim\{f \in H^2(\mathbb{C}^n) : \ \|H_{z^{k_0-1}\Phi} f\|_2 = t_0 \|f\|_2\}.$$

Next, if $d < r$, then it follows from (4.6) that

$$k_d = \min\left\{\kappa : \ \dim\{f \in H^2(\mathbb{C}^n) : \ \|H_{z^\kappa \Phi} f\|_2 = t_0 \|f\|_2\} = d(k_0 - \kappa)\right\}.$$

Similarly, we can determine the multiplicity of the index k_d, then the next largest index, *etc.* ■

Corollary 4.10. *Let $\Phi \in L^\infty(\mathbb{M}_{m,n})$. Suppose that $\|H_\Phi\|_{\mathrm{e}}$ is less than the largest nonzero superoptimal singular value of Φ. Then the indices of a monotone thematic factorization of $\Phi - \mathcal{A}\Phi$ are uniquely determined by Φ.*

Corollary 4.11. *Let $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$. Then the indices of a monotone thematic factorization of $\Phi - \mathcal{A}\Phi$ are uniquely determined by Φ.*

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